# Relativistic Chaplygin gas with field-dependent Poincaré symmetry

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February 1, 2008

**Abstract**

The relativistic generalization of the Chaplygin gas, put forward by Jackiw and Polychronakos, is derived in Duval's Kaluza-Klein framework, using a universal quadratic Lagrangian. Our framework yields a simplified proof of the field-dependent Poincaré symmetry Our action is related to the usual Nambu-Goto action [world volume] of $d$-branes in the same way as the Polyakov and the Nambu action are in strings theory.

*Lett. Math. Phys.* **57**, pp. 33-40 (2001).

## 1 Introduction

In the light-cone gauge, a relativistic $d$-brane moving in $(d+1,1)$ dimensional Minkowski space yields a $(d,1)$ dimensional isentropic and irrotational fluid, called the Chaplygin gas [1, 2]. This fluid obeys the equations of motion

$$\partial_t R + \vec{\nabla} \cdot (R\, \vec{\nabla}\, \Theta) = 0, \qquad \partial_t \Theta + \frac{1}{2}(\vec{\nabla}\, \Theta)^2 = -\frac{dV}{dR}. \tag{1.1}$$

where $R(x,t) \geq 0$ is the density, $\Theta(x,t)$ is the velocity potential, and

$$V(R) = \frac{c}{R}, \qquad c = \text{const}. \tag{1.2}$$

More generally, one can consider the polytropic potential $V = cR^n$, where $n$ is a real constants. In this Letter we shall mostly restrict ourselves to the membrane case $n = -1$.

One of the surprising features of the Chaplygin system is its large, non-linearly realized, field-dependent symmetry : its manifest $(d,1)$-dimensional

galilean symmetry extends in fact into a curious $(d+1,1)$-dimensional Poincaré dynamical symmetry [1, 2, 3, 4].

In [5], this field-dependent Poincaré symmetry was linearized by unfolding the system into a "Kaluza-Klein" spacetime $M$, obtained by adding a coordinate $s$ to non relativistic space and time, $x$ and $t$. Then all the field-dependent symmetries became Poincaré transformations of $(d+1,1)$ dimensional Minkowski space $M$ with metric $dx^2 + 2dtds$. ($t$ and $s$ are hence light-cone coordinates).

So much for the kinematics. Interestingly, the non-relativistic Chaplygin model could itself be derived by *lightlike* projection from $(d+1,1)$ dimensional Minkowski space $M$, [5][1]. Let us indeed consider two real fields $\varrho$ and $\theta$ and the potential $V(\varrho) = \lambda/\varrho$, and posit the equations (3.2) below.

Then, suitably defining the projected fields (see [5] for details) we get (1.1) with the potential (1.2). The *manifest* $(d+1,1)$-dimensional Poincaré symmetry of the higher-dimensional model can be shown, furthermore, to be preserved by the conditions (3.6), proving the dynamical Poincaré symmetry. Previous proofs use either a tedious direct calculations [3, 4] or follow by a rather tricky reduction from membrane theory [1, 6, 2].

Recently [2], Jackiw and Polychronakos presented a relativistic generalization of the Chaplygin gas, with Lagrange density

$$L^{\mathrm{JP}} = \Theta\,\partial_\tau R - \sqrt{R^2c^2 + a^2}\sqrt{c^2 + (\vec{\nabla}\Theta)^2}, \tag{1.3}$$

where $\tau$ denotes the relativistic time, $\Theta$ is the momentum potential, $\rho$ the density, and the constant $a$ is the interaction strength. This specific form is chosen so that the non-relativistic Chaplygin model is recovered in the limit $c \to \infty$. In what follows we set $c = 1$ and focus our attention to the relativistic model. The equations of motion associated to (1.3) read

$$\left\{\begin{array}{l} \partial_\tau R + \vec{\nabla}\cdot\left(\vec{\nabla}\Theta\sqrt{\dfrac{R^2+a^2}{1+(\vec{\nabla}\Theta)^2}}\right) = 0, \\[2ex] \partial_\tau\Theta + R\sqrt{\dfrac{1+(\vec{\nabla}\Theta)^2}{R^2+a^2}} = 0. \end{array}\right. \tag{1.4}$$

The manifest $(d,1)$-dimensional Poincaré symmetry of (1.3) extends, just like for its non-relativistic counterpart, to a field-dependent $(d+1,1)$-dimensional Poincaré dynamical symmetry. The additional symmetries are time reparametrization, $\widetilde{x} = x$,

$$\begin{aligned} \widetilde{\tau} &= \frac{\tau}{\cosh\omega} + \Theta(\widetilde{\tau}, x)\tanh\omega, \\ \widetilde{\Theta} &= \frac{\Theta(\widetilde{\tau}, x)}{\cosh\omega} - \tau\tanh\omega, \end{aligned} \tag{1.5}$$

[1]According to our conventions, $i, j$ are spatial indices, $\alpha, \beta, \ldots$ refer to coordinates on ordinary spacetime, and $\mu, \nu, \ldots$ refer to the extended "Kaluza-Klein" spacetime, $M$.

and space reparametrization, $\widetilde{\tau} = \tau$,

$$\begin{aligned} \widetilde{x} &= x - \hat{\gamma}\,\Theta(\tau, \widetilde{x}) \tanh\gamma + \hat{\gamma}(\hat{\gamma}\cdot x)\left(\frac{1-\cos\gamma}{\cos\gamma}\right), \\ \tilde{\Theta} &= \frac{\Theta(\tau, \widetilde{x}) - (\hat{\gamma}\cdot x)\sin\gamma}{\cosh\gamma}, \end{aligned} \tag{1.6}$$

where $\gamma = |\vec{\gamma}|$ and $\hat{\gamma} = \vec{\gamma}/\gamma$.

The aim of this Note is to derive also the relativistic model of Jackiw and Polychronakos from the *same universal model* as the non-relativistic Chaplygin system, but using *spacelike* rather than lightlike projection. First, we provide a similar interpretation of space and time reparametrizations as isometries of an extended space. Next we consider a non-linear Klein-Gordon system and point out that its “Madelung” transcription [7] yields, for a suitable choice of the potential, the universal model (3.2) and (3.5) respectively, refered to above. This will also demonstrate the field-dependent Poincaré dynamical symmetry of both Chaplygin systems.

The relation to branes is discussed in Section 4.

## 2 Unfolding

Let us start with the time reparametrizations, (1.5). Following the same recipe as in the non-relativistic case, let us add the new coordinate $\sigma = -\widetilde{\Theta}$ $\Longrightarrow \widetilde{\sigma} = -\Theta$. Then (1.5) yields $\widetilde{x} = x$,

$$\begin{aligned} \widetilde{\tau} &= \cosh\omega\,\tau - \sinh\omega\,\sigma, \\ \widetilde{\sigma} &= \cosh\omega\,\sigma - \sinh\omega\,\tau. \end{aligned} \tag{2.1}$$

which is in fact a Lorentz transformation in the $\sigma$ direction of Minkowski space with metric $-d\tau^2 + dx^2 + d\sigma^2$. ($\tau$ is hence timelike and $\sigma$ spacelike). Switching to the light-cone coordinates $t = \frac{-\tau+\sigma}{2}$, $s = \frac{\tau+\sigma}{2}$, (2.1) becomes furthermore the non-relativistic time dilation $\widetilde{x} = x$, $\widetilde{t} = e^{\delta}t$, $\widetilde{s} = e^{-\delta}s$ [5].

Space reparametrizations admit a similar interpretation. Applying again our rules, (1.6) unfolds as a rotation $d+1$-dimensional space, $\tilde{\tau} = \tau$,

$$\begin{aligned} \widetilde{x} &= x - \hat{\gamma}\sin\gamma\,\sigma - \hat{\gamma}(\hat{\gamma}\cdot x)(1-\cos\gamma), \\ \tilde{\sigma} &= \cos\gamma\,\sigma - (\hat{\gamma}\cdot x)\sin\gamma. \end{aligned} \tag{2.2}$$

Interestingly, a $(d, 1)$ dimensional Lorentz boost lifted to our extended space, $\tilde{\sigma} = \sigma$,

$$\begin{aligned} \widetilde{x} &= x + \hat{\beta}\sinh\beta\,\tau - \hat{\beta}(\hat{\beta}\cdot x)(1-\cosh\beta), \\ \tilde{\tau} &= \cosh\beta\,\tau + (\hat{\beta}\cdot x)\sinh\beta, \end{aligned} \tag{2.3}$$

($\beta = |\vec{\beta}|$, $\hat{\beta} = \vec{\beta}/\beta$) is related to the space reparametrization by the interchange of $\tau$ and $\sigma$ and by changing $\gamma$ into $i\beta$. (In the non-relativistic case, “antiboosts” and galilean boosts are related interchanging the light-cone coordinates $s$ and $t$ [5]).

## 3 Dynamics

Let us consider a Klein-Gordon field $\psi$ on $(d+1,1)$-dimensional Minkowski space,

$$\partial_\mu\,\partial^\mu\,\psi = 2\frac{d\tilde{V}}{d\psi^*} \tag{3.1}$$

where $\tilde{V} = \tilde{V}(|\psi|^2)$ is some potential. Now, in analogy with the well-known hydrodynamical transcription of non-relativistic quantum mechanics due to Madelung [7], we write $\psi = \sqrt{\varrho}\,e^{i\theta}$ to get

$$\begin{cases} \partial_\mu(\varrho\,\partial^\mu\theta) = 0, \\ \frac{1}{2}\partial_\mu\theta\partial^\mu\theta = -\dfrac{\delta V}{\delta\varrho} \end{cases} \tag{3.2}$$

where $\delta V/\delta\varrho$ is the variational derivative, and

$$V = \tilde{V} + \frac{1}{8}\frac{\partial_\mu\,\varrho\,\partial^\mu\,\varrho}{\varrho} \tag{3.3}$$

is an effective potential involving the original one, $\tilde{V}$, plus a "quantum" contribution. If we chose, following Bazeia and Jackiw [4], $\tilde{V}$ so that it cancels the second term, $\tilde{V} = V - \partial_\mu\,\varrho\,\partial^\mu\,\varrho/8\varrho$, (3.2) reduces precisely to (3.2). Similarly, the action from which the non-linear Klein-Gordon equation (3.1) is derived,

$$\int d^{d+2}x\left\{-\frac{1}{2}\,\partial_\mu\,\psi\,\partial^\mu\,\psi^* - \tilde{V}\right\} \tag{3.4}$$

is converted under the Madelung transcription into

$$\int d^{d+2}x\left\{-\frac{1}{2}\varrho\,\partial_\mu\theta\,\partial^\mu\theta - V\right\} \tag{3.5}$$

with $V$ the effective potential (3.3). The Euler-Lagrange equations are precisely (3.2). This provides a physical interpretation of the "lifted system" (3.2)–(3.5).

A word of caution, however: The upper equation in (3.2) can be viewed as a continuity equation *not* for $\varrho$ but for $(i/2)(\psi^*\,\partial_\tau\,\psi - \psi\,\partial_\tau\,\psi^*) = \varrho\,\partial^\tau\,\theta$ which, as it is well-known [8], not a positive definite expression. Thus, it can not be viewed as "particle density", only, in the best case, a "charge density". (This same interpretational problem concerns the relativistic system of Jackiw and Polychronakos (1.3) and (1.4).)

We now derive the relativistic model using instead *spacelike* projection. Let us hence consider the relativistic coordinates $x, \tau, \sigma$ on Minkowski space. Then, generalizing the rules in [5] to the relativistic context, we define the fields $\Theta$ and $\rho$ by the conditions

$$\begin{aligned} &\theta(x,\tau,-\Theta(x,\tau)) = 0, \\ &\rho(x,\tau) = \varrho(x,\tau,-\Theta(x,\tau))\,\partial_\sigma\,\theta(x,\tau,-\Theta(x,\tau). \end{aligned} \tag{3.6}$$

The first here is an implicite equation for $\Theta$, viewed as a field on ordinary space-time; once this latter has been determined, it can be used to define the projected field $\rho$. It is important to observe that this procedure is only consistent with the potential $1/\varrho$ [5]. Then the "universal" equations of motion (3.2) project, for $V \propto 1/\varrho$ only [5], to the manifestly $(d,1)$-dimensional Poincaré invariant expressions

$$\left\{\begin{array}{l} \partial_\tau(\rho\partial^\tau\Theta)+\vec{\nabla}\cdot(\rho\vec{\nabla}\Theta)=0, \\ -(\partial_\tau\Theta)^2+(\vec{\nabla}\Theta)^2+1=\dfrac{2\lambda}{\rho^2}. \end{array}\right. \tag{3.7}$$

What is the physical interpretation of these equations ? In analogy with the non-relativistic case, we would like to interpret the upper equation here as a continuity equation, i. e., a conservation equation $\partial_\alpha\, J^\alpha = 0$ for the non-relativistic current

$$J^\tau = \rho\, \partial^\tau\, \Theta, \qquad \vec{J} = \frac{\vec{\nabla}\,\Theta}{\partial^\tau\,\Theta}. \tag{3.8}$$

Now the point is that, trading $\rho$ for

$$\begin{aligned} R \equiv J^\tau &= \varrho(x,\tau,-\Theta(x,\tau))\,\partial^\tau\,\theta(x,\tau,-\Theta(x,\tau)) \\ &= \varrho(x,\tau,-\Theta(x,\tau))\,\partial_\sigma\,\theta(x,\tau,-\Theta(x,\tau))\,\partial^\tau\,\Theta, \end{aligned} \tag{3.9}$$

(3.7) become precisely the equations of Jackiw and Polychronakos in (1.4) with $a=\sqrt{2\lambda}$. In the same spirit, the universal action (3.5) becomes, under (3.6),

$$L = \frac{1}{2}\rho\left((\partial_\tau\Theta)^2-(\vec{\nabla}\Theta)^2-1\right)-\frac{\lambda}{\rho}. \tag{3.10}$$

Eliminating $\rho$ in favor of $R$ this nice quadratic expression becomes, furthermore, the square-root action (1.3).

In conclusion, we have derived the relativistic system of Jackiw and Polychronakos, [2] by spacelike projection from our universal model (3.2). (Remember that in the non-relativistic case one had to use lightlike projection). Beyond its esthetical value, our construction has the advantage that the $(d+1,1)$ dimensional dynamical Poincaré symmetry becomes a simple consequence of the manifest geometric Poincaré invariance of the universal model. This can be shown along the same lines as in [5].

A further advantage is that the conserved quantity associated to an infinitesimal Poincaré transformation $(X^\mu)$ of $M$ is readily found using the [symmetric] energy-momentum tensor of (3.5) constructed in [5],

$$\begin{aligned} &Q = \int \frac{T^\tau{}_\mu X^\mu}{\partial_\sigma\theta}\, d^dx, \\ &T_{\mu\nu} = -\varrho\,\partial_\mu\theta\partial_\nu\theta + g_{\mu\nu}\left(\tfrac{1}{2}\rho\partial_\omega\theta\partial^\omega\theta+V(\varrho)\right). \end{aligned} \tag{3.11}$$

These formulae yield

$$
\begin{aligned}
&\mathcal{H} = \tfrac{1}{2}\rho[(\partial_\tau\Theta)^2 + (\vec{\nabla}\Theta)^2 + 1] + \frac{\lambda}{\rho}, && \text{energy}\\
&\mathcal{P}_i = -\rho\partial_i\Theta\,\partial_\tau\Theta, && \text{momentum}\\
&\mathcal{N} = -\rho\partial_\tau\Theta, && \text{relat. ``number''}\\
&\mathcal{D} = \mathcal{H}\Theta + \mathcal{N}\tau, && \text{time reparametrization}\\
&\mathcal{G}_i = x_i\mathcal{N} + \Theta\mathcal{P}_i && \text{space reparametrization}
\end{aligned}
\tag{3.12}
$$

which become, inserting $R$, exactly the conserved quantities in [2].

Somewhat paradoxically, both *relativistic* systems, (1.3) and (3.10), are also Galilei-invariant, simply because the $(d, 1)$ dimensional Galilei group is a subgroup of the Poincaré group in $(d + 1, 1)$ dimensions. Applying our rules backwards, for a galilean boost we get, e. g., the field-dependent action

$$
\begin{aligned}
\widetilde{x} &= x - \tfrac{1}{2}\alpha\tau - \tfrac{1}{2}\alpha\widetilde{\Theta},\\
\widetilde{\tau} &= (1 + \tfrac{1}{4}\alpha^2)\tau - \alpha\cdot x + \tfrac{1}{4}\alpha^2\widetilde{\Theta},\\
\Theta &= \widetilde{\Theta}(1 + \tfrac{1}{4}\alpha^2) + \alpha\cdot x - \tfrac{1}{4}\alpha^2\tau + \widetilde{\Theta}(1 + \tfrac{1}{4}\alpha^2).
\end{aligned}
\tag{3.13}
$$

## 4 Relation to d-branes

Our framework here is closely related to the so-called non-parametric representation of $d$ branes [6]. Our "vertical" variable $\sigma$ (*alias* the field $-\Theta$) is in fact the $z$ coordinate of the $d$-brane propagating in $(d+1, 1)$ dimensional Minkowski space, and our "lifted" field $\theta$ is (minus) their $u$, the function whose level sets describe the $d$-brane as $\theta(x, \tau, z(x, \tau)) = 0$, — which is our first condition in (3.6).

In terms of $\theta$, the motion of the $d$-brane is governed by the action

$$\int\sqrt{\partial_\mu\theta\,\partial^\mu\theta}\,d^{d+2}x, \tag{4.1}$$

whose equations of motion read

$$\partial_\mu\left(\frac{\partial^\mu\theta}{\sqrt{\partial_\nu\theta\,\partial^\nu\theta}}\right) = 0. \tag{4.2}$$

The integrand here is in fact the "Nambu" world volume of the $d$-brane [9],

$$\sqrt{\partial_\mu\theta\,\partial^\mu\theta} = \sqrt{\det(G_{\alpha\beta})}, \qquad G_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu. \tag{4.3}$$

The point is that one can get rid of the square root, just like for a free relativistic particle. (This latter can be described either by the usual invariant length action $-m\int\sqrt{-\dot{x}^2}\,d\tau$, or by a quadratic action plus a constraint, when an auxiliary variable is added [10]). Let us hence enlarge our

pure scalar theory involving $\theta$ alone by introducing an auxiliary field we call $\rho$. Then (4.2) is readily seen to imply the first equation in (3.2); but both equations (3.2) derive from our quadratic Lagrangian (3.5). Conversely, inserting $\rho$ into our action and equations of motion, (4.1) and (4.2) are recovered. (The two-dimensional analog is string theory, where the quadratic Polyakov action can be used instead of the Nambu-Goto expression [11, 10]).

# 5 Discussion

While the Lagrangian (1.3) is first-order in the time derivative and the Hamiltonian $\sqrt{R^2+a^2}\sqrt{\left(\vec{\nabla}\Theta\right)^2+1}$ contains ugly square roots, our expressions are quadratic, as in ordinary relativistic scalar field theory. The two expressions are equivalent; our quadratic expression could have some advantage when the quantization of the system is considered. Let us emphasise that this approach not only considerably simplifies the proof of the dynamical Poincaré symmetry, but also explains its rather mysterious origin. The possibility of having differently-looking but still equivalent systems corresponds to the freedom of chosing the kinetic term [2].

The strange fact recognized by Jackiw and Polychronakos is that the the non-relativistic Chaplygin gas is simultaneously the $c\to\infty$ limit, and also equivalent to their relativistic model [2]. This can also be seen in our framework : deforming the space-like fibration into lightlike amounts, on the one hand, to taking the non-relativistic limit [12]. On the other hand, (3.6) is merely the definition of the projected fields and does not impose any restriction. The two systems are hence equivalent through the universal model.

Let us mention, in conclusion, that our formalism can also be used to study the conformal properties of gas dynamics [13]. For the adiabatic potential $V(\varrho)\propto\varrho^n$, the action (3.5) is readily seen to be invariant w. r. t. the $(d+1,1)$ dimensional conformal group $\O(d+1,2)$ if and only if the polytropic exponent is

$$n=1+\frac{2}{d}. \tag{5.1}$$

(This can also be seen from the trace condition $T^\mu_{\ \mu}=0$ of the energy-momentum tensor (3.11)).

In the free case, $\O(d+1,2)$ is a [field-dependent] symmetry also for the reduced system [5]. For $V\neq0$, however, the potential is only consistent with equivariance,

$$\begin{aligned}\partial_\sigma\varrho=0 \quad&\Longrightarrow\quad \varrho=\rho(x,\tau),\\ \partial_\sigma\theta=1 \quad&\Longrightarrow\quad \theta=\Theta(x,\tau)+\sigma,\end{aligned} \tag{5.2}$$

rather than with the generalized condition (3.6). Equivariance reduces, however, the $(d+1,1)$ dimensional conformal symmetry to its mere [(d,1)-

Poincaré]$\times\mathbf{R}$ subgroup, the $\mathbf{R}$ representing the vertical translations, whose associated conserved quantity is the "number" $N$. Let us recall that in the non-relativistic case the corresponding subgroup is the $(d, 1)$ dimensional Schrödinger group [14, 5, 13].

Acknowledgements. We are indebted to Professors C. Duval, G. Gibbons, R. Jackiw and J.-W. van Holten for their interest, and to J. Hoppe for discussions.

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